Blackbody radiation

According to thermodynamics, a black body is an object that absorbs light (electromagnetic radiation) of all frequencies and also radiates light of all frequencies when it is kept at a given temperature. The distribution of density of radiated energy for various frequency of light was still an unsolved problem at the end of nineteenth century. The distribution of energy density ($\mathcal{E}$) over frequency ν, as measured in experiments is given by the plot below.

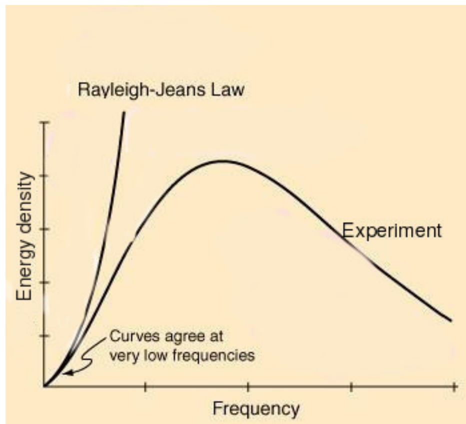


Figure 1: Plot of energy density $\mathcal{E}$ against frequency ν

This was obtained by measuring the energy density of the radiation emerging from spherical cavity of the type shown below. The energy density of the radiation emerging out of the small opening in the cavity is measured for various frequencies to obtain the plot shown above.

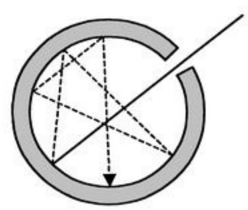


Figure 2: A spherical cavity with a tiny opening can act as a black body (Source : web.mit.edu)

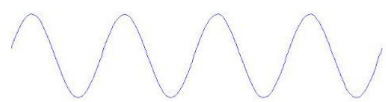
At the time when this result of experiment was available, it was thought that light of a given frequency was an electromagnetic wave. If that were the case, the electromagnetic waves inside the cavity would have formed standing wave patterns inside the spherical cavity. Using this idea, Rayleigh and Jeans tried to give an explanation for the $d\mathcal{E}$ vs. ν plot of the experiment data shown in fig (1) above. They got the expression for the number of standing waves per unit volume (ie., the density of standing wave) in a frequency range between ν and $\nu + d\nu$ to be

$$\frac{8\pi\nu^2}{c^3}d\nu \tag{1}$$

Each standing wave in the above cavity can be thought of as a simple harmonic oscillator. From kinetic theory, we know that the average thermodynamic internal energy of an oscillating particle at absolute temperature T (in Kelvins) is $k_B T$. Half of this quantity is its average kinetic energy and the other half is the average potential energy. By multiplying the density of standing waves with the energy of each standing wave we get the energy density out to be

$$d\mathcal{E} = \frac{8\pi\nu^2(k_B T)}{c^3} d\nu \tag{2}$$

If we plot this $d\mathcal{E}$ as a function of frequency ν we get the monotonically increasing curve in fig (1). By comparing the plots in fig(1), it is clear that the Rayleigh-Jeans expression fails to explain the experimentally obtained plot of energy density. However it is worthy to note that at low values of frequency the plots coincide.



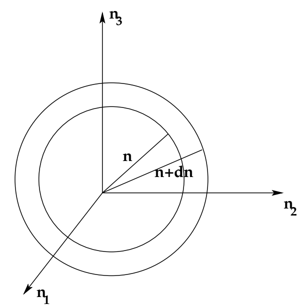


Figure 3:

Counting of standing wave modes

Standing waves are formed when a wave interferes with its own reflected wave constructively. A simple example is the standing wave on a string tied at both ends. Constructive interference will happen if the nodes of the wave are at end points of the string. This can happen if the wavelength λ is such that the length of the string L is

$$L = m\frac{\lambda}{2} \tag{3}$$

where $m = 1, 2, 3, \ldots$ gives us the number of waves of wavelengths λ. Standing wave mode with $m = 8$ is shown in the figure on the left in fig(3).
In the above case, the displacement of the point x on the string at time t has the form

$$A\sin(2\pi\nu t - 2\pi m x) + A\sin(2\pi\nu t + 2\pi m x) = 2A\ \sin 2\pi m x\ \cos 2\pi\nu t$$

This represents a *standing* wave (as opposed to a wave that travels or propagates).

Counting the number of waves requires us to find m for a standing wave. From fig(2), it is clear how we find m. For $m = 1$ we can see one full oscillation of a wave of wavelength $\lambda = L$ as we move from the position $x = 0$ to $x = L$. Wavenumber m of the standing

waves in the spherical cavity used in black body radiation experiment have two important features. It is easy to see that standing waves can form only along the diameters of sphere. This is so because, inside a sphere, *an onward wave and its reflected waves can travel along the same line only in the radial directions.* Standing waves will form for those wavelengths for which the diameter a of the sphere is an integral multiple of wavelength ie., $a = m\frac{\lambda}{2}$. The *number of waves per unit length* n turns out to be $\frac{m}{2a} = \frac{1}{\lambda} = n$. In the context of black body radiation, the radius a of the sphere ($\sim 5 \times 10^{-2} m$) will be bigger than the wavelength of light ($\sim 5 \times 10^{-7} m$) by several orders of magnitude. The number m of standing waves formed will be so large ($\sim 10^5$) that n can be treated as a continuous variable. This is the first feature.
As we can draw an infinite number of diameters for a sphere, the standing waves can form along each of these diameters. Wavenumbers along each of these radii could be different, and thus the wavenumber per unit length has to be a vector, $\mathbf{n} = n_1\hat{\mathbf{x}} + n_2\hat{\mathbf{y}} + n_3\hat{\mathbf{z}}$. This is the second feature. We will call this *wavevector* and henceforth denote the modulus of this vector using $n = |\mathbf{n}|$. It is clear that $n = \sqrt{n_1^2 + n_2^2 + n_3^2}$, we can find out the number (or amount) of waves contained in an infinitesimal interval $(n, n + dn)$. From the figure on the right in fig(3) we understand that this is just the number of waves contained in a spherical shell in $\mathbf{n}$-space that has an inner radii n and an outer radii $n + dn$.

Normally, the number of modes contained in such a shell would be the surface area of the inner sphere times the thickness of the shell, ie., $4\pi n^2 dn$. However electromagnetic waves have two possible polarizations, so the total number of waves will be twice the number above, giving us $8\pi n^2 dn$. But $n = \frac{1}{\lambda}$ and $\lambda = \frac{c}{\nu}$. Since electromagnetic waves have the same velocity in vacuum c is a constant. Thus the number of standing wave modes *per unit volume* in the wavenumber interval $(n, n + dn)$ comes out to be the expression in eqn(1)

$$8\pi n^2 dn = 8\pi \frac{\nu^2}{c^3} d\nu$$

Using the statistical distribution due to Boltzmann, known at the time, and making an assumption that the energy of an electromagnetic wave of frequency ν is

$$E = nh\nu \tag{4}$$

(where h is a constant and $n = 0, 1, 2, \ldots$ any non-negative integer) Max Planck arrived at an expression for the energy density

$$d\mathcal{E} = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{\exp\left(\frac{h\nu}{k_B T} - 1\right)} \tag{5}$$

The value of the *Planck's constant* h that correctly reproduced the experimental curve was $h = 6.626 \times 10^{-34}$ Joules sec. The plot of the distribution in eqn(4). above matched with the experimental plot in fig(1). For low values of frequency Planck's distribution reduces the the Rayleigh-Jeans distribution which agrees with the observation made earlier that Rayleigh-Jeans distribution coincides with experimental curve for low frequencies.
At that time, there was no experimental justification known for the assumption $E = nh\nu$ made in the derivation. It seemed to indicate that the energy of an electromagnetic wave of frequency ν has to be in quanta (multiples) of a basic unit $h\nu$.