Waves

de Broglie's hypothesis was supported by the data of Davisson and Germer experiment. However if we look at the nature of the de Broglie waves it is found to be not a wave of definite wavelength and frequency. de Broglie waves behave more like wave groups that consists of several waves, each with a wavelength and frequency that is different from the remaining. To understand this better, let us learn about waves in some detail.
Waves occur when a disturbance at a location at some instant of time is determined by the disturbance at another position at an earlier time. An example would be the waves that we see on the surface of water. We could see that the displacement of water surface at time t at position $\mathbf{r}$ is the same as the one at another position $\mathbf{r}'$ at an earlier time t_0. Displacement $y(x,t)$ (or disturbance) of a wave that travels at a velocity v along the X-axis satisfy the wave equation

$$\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = 0 \tag{1}$$

Solution to the above equation has the form

$$y(x,t) = f(x \pm vt)$$

The wave represented by $f(x - ct)$ progresses along the positive X-axis and $f(x + vt)$ along the negative X-axis.
Sinusoidal waves have displacement of the form $y(x,t) = A\sin(kx - \omega t)$ or $y(x,t) = A\cos(kx - \omega t)$. The constant A is the amplitude of the wave. A snapshot of the wave at any instant ($t = 0$ here) will look like the one in fig(1) The distance between successive

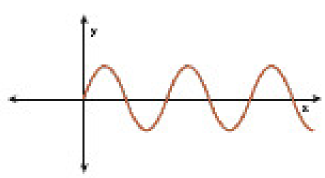

Figure 1: Spatial distribution of wave at time t=0.

maxima (minima) is the wavelength $\lambda = \frac{2\pi}{k}$ of the wave. The number of oscillations of any point per unit time is the frequency $\nu = \frac{\omega}{2\pi}$ of the wave. Velocity of the wave is

$$v = \frac{\omega}{k} \tag{2}$$

Most of the waves that we are familiar with, do not have a definite wavelength and frequency. This is because they are formed by superposing (adding) waves of different wavelengths and frequencies. Let us take up the case of the simplest such superposition of two waves that differ in wavelength and frequencies with same amplitude. Adding the two we get

$$A\ \sin(k_1 x - \omega_1 t) + A\ \sin(k_2 x - \omega_2 t) = 2A\ \sin\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\ \cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)$$

The resultant expression looks like a wave certainly. But one can say its wavelength and frequency looks ambiguous. The sine term on the right hand side looks like a wave of wavelength $\frac{k_1+k_2}{2}$ and frequency $\frac{\omega_1+\omega_2}{2}$. What multiplies it on the RHS must be its amplitude, namely,

$$2A\ \cos\left(\frac{k_1-k_2}{2}x-\frac{\omega_1-\omega_2}{2}t\right)$$

The amplitude itself looks like a cosine wave of wavelength $\frac{k_1-k_2}{2}$ and frequency $\frac{\omega_1-\omega_2}{2}$. In other words, the resultant wave is one whose amplitude is also a wave. The snapshot of the resultant wave is given in fig(2) below. The two constituent waves of this wave group travels

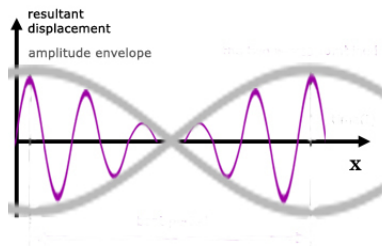

Figure 2: Spatial distribution of wave beat at time t=0. Thin curve is the actual wave of wavelength $\frac{k_1+k_2}{2}$ and frequency $\frac{\omega_1+\omega_2}{2}$, whose amplitude is varying. The thick curve represents the envelop that shows the change of amplitude of the former.

at different velocities $v_1 = \frac{\omega_1}{k_1}$ and $v_1 = \frac{\omega_2}{k_2}$ respectively. The superposition will eventually disappear as the constituent waves will separate away from each other. This property is called the dispersion. A famous example of dispersion is that of light as it passes through a prism. Dispersion occurs because in glass light of different wavelengths have different velocities. Dispersive property of the above wave group could be measured using the quantity

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\omega_1-\omega_2}{k_1-k_2} \tag{3}$$

called the *group velocity* of the wave group. If ω and k take a continuous range of values the group velocity is

$$v_g = \frac{d\omega}{dk} \tag{4}$$

In contrast, the velocity of a wave with definite wavelength and frequency as in eqn(2) is called the *phase velocity*. It is found that be Broglie waves could only be wave groups made of different waves. This leads to an interesting consequence called the uncertainty principle.