Waves

According to de Broglie's proposal, a particle with mass m that is located somewhere with some momentum must have a de Broglie wave associated with it. However, we know that it is difficult to find the position of that particle if one is far away from the particle. Thus the de Broglie wave must be localized around the position of that particle. Let us consider a de Broglie wave that is extremely localized at position $x = x_0$ as shown in fig(1).

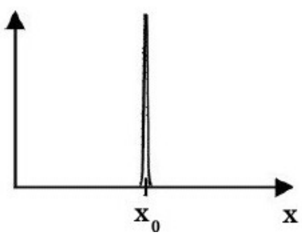

Figure 1: de Broglie wave localized at position $x = x_0$

It is a well known result from Fourier series that if one superposes cosine waves with frequencies that are integral multiples of a basic frequency, all with same amplitude, one gets a sharp pulse at a position that is decided by the phase of the cosine waves. Fig(2) provides the plot of

$$\sum_{m=0}^{n} \cos(\frac{mx}{T}) \tag{1}$$

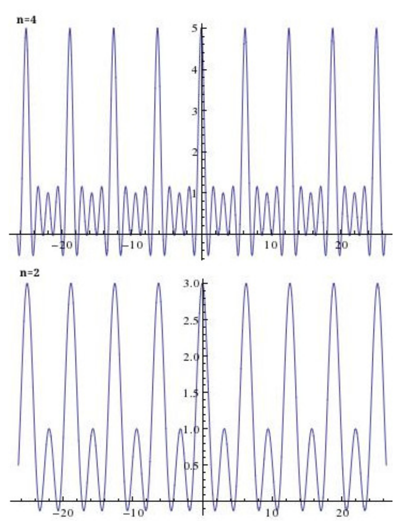

Figure 2: Plot of $\sum_{m=0}^{n} \cos(\frac{mx}{T})$ for $n = 2, 4$

Notice that as we increase the number of waves that are superposed, one of the peaks grow taller than the rest.

Thus, if we include waves of all possible frequencies by taking n to be a large number and also let the period T to be a large number, the resultant wave would look exactly like the one in fig(1). However in this limit, n becomes a continuous variable, usually denoted by k, that takes values in the range $0 \leq k \leq \infty$. In this limit, eqn(1) becomes [1]

$$\int_0^\infty dk \ \cos kx \tag{2}$$

The take home lesson from this is that if one wants to localize a wave at a position we need to superpose a large number of waves with same amplitude. For a perfectly localized wave infinite number of waves need to be superposed. The better a wave is localized the less certain is its wavenumber k. (For a perfectly localized wave the wavenumber is completely uncertain).
But according to de Broglie, the momentum of a particle is

$$p = \frac{h}{\lambda} = \frac{h \times 2\pi}{2\pi\lambda} = \hbar k \tag{3}$$

If k is uncertain the momentum of that particle must be uncertain by a proportional amount. This fact is stated in a quantitative form by German physicist Werner Heisenberg, since then called Heisenberg's *uncertainty principle*, as follows:

For a particle, the uncertainty in measurement of its position Δx and and the uncertainty in measurement of its momentum Δp obey

$$\Delta x \ \Delta p \geq \frac{\hbar}{2} \tag{4}$$

Thus it is impossible to make simultaneous measurements of position and momentum with arbitrarily small accuracy.

This relation puts a fundamental limitation on the precisions involved in the measurement of position and momentum for any physical system.

What is the uncertainty (also termed *error* or *accuracy*) in the measurement of a quantity? How do we find it? Uncertainty tells us how much is our "lack of knowledge" about the measured value of the quantity. In every experiment, one makes multiple measurements to find the value of the quantity of interest. A representative value for the quantity is chosen by statistical methods like, for instance, mean or median or mode. If we choose mean as the value of the quantity, we could find what the average deviation from the mean is. This is the standard deviation of our data which is a measure of how much uncertain we are about our value which is the mean. In this case we get standard deviation as uncertainty in the quantity measured. The result of the experiment is always stated as the value of the quantity along with the error bar. For instance, the value of position of a particle could be stated as $x \pm \Delta x$ where Δx is the error in the measurement.

[1] The integral in eqn(1) is the Fourier representation of the Dirac delta distribution.